

# Modified frequentist determination of confidence intervals and its equivalence to the Bayesian method

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## Abstract

We propose modified frequentist definition for the determination of confidence intervals for the case of Poisson statistics. Namely, we require that  $1 - \beta' \geq \sum_{n=o}^{n_{obs}+k} P(n|\lambda) \geq \alpha'$ . We show that this definition is equivalent to the Bayesian method with prior  $\pi(\lambda) \sim \lambda^k$ . We also propose modified frequentist definition for the case of nonzero background.

In high energy physics one of the standard problems [1] is the determination of the confidence intervals for the parameter  $\lambda$  in Poisson distribution

$$P(n|\lambda) = \frac{\lambda^n}{n!} \exp(-\lambda) \quad (1)$$

from the observed number  $n_{obs}$  of events. There are two methods to solve this problem - the frequentist and the Bayesian. In Bayesian method [1, 2] due to Bayes theorem the probability density for the  $\lambda$  parameter is determined as

$$p(\lambda|n_{obs}) = \frac{P(n_{obs}|\lambda)\pi(\lambda)}{\int_0^\infty (P(n_{obs}|\lambda')\pi(\lambda'))d\lambda'} \quad (2)$$

Here  $\pi(\lambda)$  is the prior function and in general it is not known that is the main problem of the Bayesian method. Formula (2) reduces the statistics problem to the probability problem. At the  $(1 - \alpha)$  probability level the parameters  $\lambda_{up}$  and  $\lambda_{down}$  are determined from the equation <sup>1</sup>

$$\int_{\lambda_{down}}^{\lambda_{up}} p(\lambda|n_{obs})d\lambda = 1 - \alpha \quad (3)$$

and the unknown parameter  $\lambda$  lies between  $\lambda_{down}$  and  $\lambda_{up}$  with the probability  $1 - \alpha$ . The solution of the equation (4) is not unique. One can define

$$\int_{\lambda_{up}}^\infty p(\lambda|n_{obs})d\lambda = \alpha' \quad (4)$$

$$\int_0^{\lambda_{down}} p(\lambda|n_{obs})d\lambda = \beta' \quad (5)$$

In general the parameters  $\alpha'$  and  $\beta'$  are arbitrary except the evident equality

$$\alpha' + \beta' = \alpha \quad (6)$$

The most popular are the following options [1]:

1.  $\lambda_{down} = 0$  - upper limit.
2.  $\lambda_{up} = \infty$  - lower limit.
3.  $\int_0^{\lambda_{up}} p(\lambda|n_{obs})d\lambda = \int_{\lambda_{down}}^\infty p(\lambda|n_{obs})d\lambda = \frac{\alpha}{2}$  - symmetric interval.
4. The shortest interval -  $p(\lambda|n_{obs})$  inside the interval is bigger or equal to  $p(\lambda|n_{obs})$  outside the interval.

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<sup>1</sup>Usually  $\alpha$  is taken equal to 0.05.

In frequentist approach the Neyman belt construction [3] is used for the determination of the confidence intervals. Namely, for each  $\lambda$  we require that

$$\sum_{n=n_-(\lambda)}^{n_+(\lambda)} P(n|\lambda) \geq 1 - \alpha. \quad (7)$$

For the observed number of events  $n_{obs}$  the equation (7) allows to determine the confidence interval  $[\lambda_{down}(n_{obs}), \lambda_{up}(n_{obs})]$  of possible parameters  $\lambda$ . Note that as in Bayesian approach the choice of  $n_+(\lambda)$  and  $n_-(\lambda)$  is not unique. In general we have the following equations [4, 5, 6] for the determination of  $\lambda_{down}$  and  $\lambda_{up}$  :

$$\sum_{n=n_{obs}}^{\infty} P(n, \lambda_{down}) = \beta', \quad (8)$$

$$\sum_{n=0}^{n_{obs}} P(n, \lambda_{up}) = \alpha', \quad (9)$$

where

$$\alpha' + \beta' = \alpha. \quad (10)$$

In this paper we show that the modified frequentist definition of the confidence interval is equivalent to the Bayesian approach. Consider the probability to observe the number of events  $n \leq n_{obs}$

$$P_-(n_{obs}|\lambda) = \sum_{n=0}^{n_{obs}} P(n|\lambda). \quad (11)$$

To determine possible values  $\lambda_{down}$  and  $\lambda_{up}$  of the confidence interval we require that

$$1 - \beta' \geq P_-(n_{obs}|\lambda) \geq \alpha', \quad (12)$$

where  $\alpha' + \beta' = \alpha$ . The equations for the determination  $\lambda_{up}$  and  $\lambda_{down}$  have the form

$$P_-(n_{obs}|\lambda_{up}) = \alpha', \quad (13)$$

$$P_-(n_{obs}|\lambda_{down}) = 1 - \beta'. \quad (14)$$

Note that as in the case of Bayesian approach the choice of  $\alpha'$  and  $\beta'$  is not unique. Due to the identity [6]

$$P_-(n_{obs}|\lambda) = \int_{\lambda}^{\infty} P(n_{obs}|\lambda') d\lambda' \quad (15)$$

the confidence interval  $[\lambda_{down}, \lambda_{up}]$  is determined from the equations

$$\alpha' = \int_{\lambda_{up}}^{\infty} P(n_{obs}|\lambda') d\lambda', \quad (16)$$

$$\beta' = \int_0^{\lambda_{down}} P(n_{obs}|\lambda') d\lambda'. \quad (17)$$

The parameter  $\lambda$  lies in the interval

$$\lambda_{down} \leq \lambda \leq \lambda_{up} \quad (18)$$

with the probability  $(1 - \alpha' - \beta')$ . Due to equations (16,17) our modified frequentist definition (12) is equivalent to Bayes definitions (3,4,5) with flat prior  $\pi(\lambda) = 1$ , namely:

$$\int_{\lambda_{down}}^{\lambda_{up}} P(n_{obs}|\lambda') d\lambda' = 1 - \alpha' - \beta'. \quad (19)$$

The coverage of the definition (12) means the following. For a hypothetical ensemble of similar experiments the probability to observe the number of events  $n \leq n_{obs}$  satisfies the inequalities (12). As we noted before the choice of  $\lambda_{down}$  and  $\lambda_{up}$  is not unique. Probably the most natural choice is the use of the ordering principle. According to this principle we require that the probability density  $P(n_{obs}|\lambda)$  inside the confidence interval  $[\lambda_{down}, \lambda_{up}]$  is bigger or equal to the probability density outside this interval. For Poisson distribution this requirement leads to the formula

$$P(n_{obs}|\lambda_{down}) = P(n_{obs}|\lambda_{up}) \quad (20)$$

for the determination of  $\lambda_{up}$  and  $\lambda_{down}$ . For such ordering principle  $\alpha'$  and  $\beta'$  are not independent quantities. It is natural to use  $\alpha = \alpha' + \beta'$  as a single free parameter. Note that the equations (9) and (13) for the determinations of an upper limit  $\lambda_{up}$  in frequentist and modified frequentist approach coincide whereas the equations (8) and (14) are different. Namely, the equation (14) is equivalent to the equation

$$\sum_{n=n_{obs}+1}^{\infty} P(n|\lambda) = \beta'. \quad (21)$$

So we see that the summation in our modified equation (21) starts from  $n_{obs} + 1$  whereas in classical frequentist equation (8) the summation starts from  $n_{obs}$ . Classical frequentist equation (8) is equivalent to the Bayes equation (5) with prior  $\pi(\lambda) \sim \frac{1}{\lambda}$ .

Note that for the case of continuous variable  $x$  ( $+\infty > x > -\infty$ ) our inequality (12) takes the form

$$1 - \beta' \geq \int_{-\infty}^{x_{obs}} f(x'|\lambda) dx' \geq \alpha' . \quad (22)$$

Here  $f(x, \lambda)$  is the probability density function and  $\lambda$  is some unknown parameter. The equations for the determination of  $\lambda_{up}$  and  $\lambda_{down}$  are

$$\int_{-\infty}^{x_{obs}} f(x'|\lambda_{up}) dx' = \alpha' , \quad (23)$$

$$\int_{x_{obs}}^{\infty} f(x'|\lambda_{down}) dx' = \beta' . \quad (24)$$

The equations (23,24) coincide with classical Neyman belt equations.

It is possible to generalize our modified frequentist definition (12), namely:

$$1 - \beta' \geq P_{-}(n_{obs}|\lambda; k) \geq \alpha' , \quad (25)$$

where

$$P_{-}(n_{obs}|\lambda; k) \equiv \sum_{n=0}^{n_{obs}+k} P(n|\lambda) \quad (26)$$

and  $k = 0, \pm 1, \pm 2, \dots$

One can find that formulae (25,26) lead to Bayes equations (4,5) with the prior function  $\pi(\lambda) \sim \lambda^k$ . We can further generalize formulae (25,26) by the introduction

$$P_{-}(n_{obs}|\lambda; c_k) \equiv \sum_k c_k^2 P_{-}(n_{obs}|\lambda; k) , \quad (27)$$

where  $\sum_k c_k^2 = 1$ . Again we require that

$$1 - \beta' \geq P_{-}(n_{obs}|\lambda; c_k) \geq \alpha' . \quad (28)$$

One can find that our definition (28) is equivalent to Bayes approach with prior function

$$\pi(\lambda) = \sum_k c_k^2 l_k \lambda^k , \quad (29)$$

where

$$l_k = \frac{n!}{(n+k)!} . \quad (30)$$

For the case when we have nonzero background the parameter  $\lambda$  is represented in the form

$$\lambda = b + s. \quad (31)$$

Here  $b \geq 0$  is known background and  $s$  is unknown signal. In Bayes approach the generalization of the formula (2) reads

$$p(s|n_{obs}, b) = \frac{P(n_{obs}|b+s)\pi(b, s)}{\int_0^\infty (P(n_{obs}|b+s')\pi(b, s')ds')}. \quad (32)$$

For flat prior we have

$$p(s|n_{obs}, b) = \frac{P(n_{obs}|b+s)}{\int_b^\infty (P(n_{obs}|\lambda')d\lambda')}. \quad (33)$$

So we see that the main effect of nonzero background is the appearance of the factor

$$K(n_{obs}, b) = \int_b^\infty P(n_{obs}|\lambda')d\lambda' \quad (34)$$

in the denominator of formula (33). For zero background  $K(n_{obs}, b=0) = 1$ . One can interpret the appearance of additional factor  $K(n_{obs}, b)$  in terms of conditional probability. Really, for flat prior the  $P(n_{obs}, \lambda)d\lambda$  is the probability that parameter  $\lambda$  lies in the interval  $[\lambda, \lambda+d\lambda]$ . For the case of nonzero background  $b$  parameter  $\lambda = b+s \geq b$ . The probability that  $\lambda \geq b$  is equal to  $p(\lambda \geq b|n_{obs}) = K(n_{obs}, b)$ . The conditional probability that  $\lambda$  lies in the interval  $[\lambda, \lambda+d\lambda]$  provided  $\lambda \geq b$  is determined by the standard formula of the conditional probability

$$p(\lambda, n_{obs}|\lambda \geq b)d\lambda = \frac{p(\lambda, n_{obs})}{p(\lambda \geq b)}d\lambda = \frac{p(\lambda, n_{obs})}{K(n_{obs}, b)}d\lambda \quad (35)$$

and it coincides with the Bayes formula (33).

In the frequentist approach the naive generalization of the inequality (12) is

$$1 - \beta' \geq P_-(n_{obs}|s+b) \geq \alpha'. \quad (36)$$

One can show that

$$1 - \alpha' - \beta' = \int_{b+s_{down}}^{b+s_{up}} P(n_{obs}|\lambda')d\lambda' \leq \int_b^\infty P(n_{obs}|\lambda')d\lambda'. \quad (37)$$

However the main drawback of the definition (36) is that the probability that the signal  $s$  lies in the interval  $0 \leq s \leq \infty$  is equal to  $\int_b^\infty P(n_{obs}|\lambda')d\lambda'$  and it is less than unity for

nonzero background  $s > 0$  that contradicts to the intuition that the full probability that the signal  $s$  lies between zero and infinity must be equal to unity. To cure this drawback let us require that

$$1 - \beta' \geq \frac{P_{-}(n_{obs}|s+b)}{P_{-}(n_{obs}|b)} \geq \alpha' . \quad (38)$$

The inequality (38) leads to the equations for the determination of  $s_{down}$  and  $s_{up}$  which coincide with the corresponding Bayes equations. The generalization of the inequalities (38) is straightforward, for instance the inequality (28) reads

$$1 - \beta' \geq \frac{P_{-}(n_{obs}|b+s; c_k)}{P_{-}(n_{obs}|b; c_k)} \geq \alpha' . \quad (39)$$

Upper limit on the signal  $s$  derived from the inequality (38) coincides with the upper limit in  $CL_s$  method [7, 8].

Note that frequentist equations (8,9) for  $\lambda_{up} = \lambda_{down}$  don't satisfy the evident equality  $\alpha' + \beta' = 1$ . The natural generalization of the equations (8,9) looks as follows

$$P_{-1}(n_{obs}|\lambda_{up}) = \alpha' , \quad (40)$$

$$P_{+1}(n_{obs}|\lambda_{down}) = \beta' , \quad (41)$$

where

$$P_{-1}(n_{obs}|\lambda) = \sum_{n=0}^{n_{obs}-1} P(n|\lambda) + \frac{1}{2}P(n_{obs}|\lambda) , \quad (42)$$

$$P_{+1}(n_{obs}|\lambda) = \sum_{n=n_{obs}+1}^{\infty} P(n|\lambda) + \frac{1}{2}P(n_{obs}|\lambda) . \quad (43)$$

Note that

$$P_{-1}(n_{obs}|\lambda) + P_{+1}(n_{obs}|\lambda) = 1 \quad (44)$$

and  $\alpha' + \beta' = 1$  for  $\lambda_{up} = \lambda_{down}$ . The equations (40,41) are equivalent to the Bayes equations (4,5) with prior

$$\pi(\lambda) = \frac{1}{2} \left( 1 + \frac{n_{obs}}{\lambda} \right) . \quad (45)$$

The modified frequentist definition (12) takes the form

$$1 - \beta' \geq P_{-1}(n_{obs}|\lambda) \geq \alpha' . \quad (46)$$

To conclude let us stress our main result. For Poisson distribution we have proposed modified frequentist definition of the confidence interval and have shown the equivalence of the modified frequentist approach and Bayes approach.

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